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JUSTIFICATION OF THE FAST MULTIPOLE METHOD FOR THE STOKES SYSTEM IN THE CASE OF THE INTERIOR DIRICLET PROBLEM

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We provide all necessary theoretical statements concerning the hydrodynamical double layer potential, which enable the application of an adaptive version of the fast multipole method of Greengard and Rokhlin to the interior Dirichlet problem of the Stokes system and present numerical experiments confirming these theoretical statements. Bibliography: 26 *titles. Illustrations*: 3 *figures.*

1 Introduction

The adaptive fast multipole method (AFMM) has proven to be extremely efficient and successful for solving problems of mathematical physics and industrial engineering. In particular, it has been applied to the Laplace [1]–[4], wave [5], Helmholtz [6]–[8], and Maxwell [9]–[11] equations, as well as to the linear elastostatics [12]–[14]. Concerning the Stokes equations, well known from linear hydrodynamics (cf. [15]–[20]), up to now there exist some contributions in the framework. In [21], Greengard et al. presented a solution procedure based on the fast multipole method, which uses Sherman's complex variable formulation for the biharmonic boundary value problems with the Stokes flow as a special case. Peault et al. applied in [22] a fast multipole formulation to the direct and indirect formulations of boundary integral equations for the two-dimensional Stokes cavity flow. They used mixed multipolar expansion of kernel in real variables directly and showed that the indirect formulation in the form of boundary integral equations with double layer potential is more stable and leads to better conditioned systems of equations. Khoromskij et al. [23] presented an alternative technique of almost linear complexity for solving elliptical partial differential equations based on their reduction to the interface. The proposed approach essentially deals with interface equations and inherits many beneficial features of FEM and BEM. In the case of Stokes equation, the interface reduction method is based on either the stream function-vorticity formulation or the use of the special Poincar´-Steklov operator.

Unlike another works on fast methods for the Stokes problem, our method is based on the classical hydrodynamical potential theory (cf. [24]): With help of the Stokes fundamental tensor, we derive a representation of the solution that depends on boundary integrals with an unknown vector-valued source density. This source density can be determined as a unique solution of a

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system of boundary integral equations. For the discretization of this system we use a boundary element formulation based on the Nyström method. The resulting linear algebraic system of equations is of the form

$$\left(\sum_{j=1}^{m} \mathbf{k}(\mathbf{x}^i, \mathbf{x}^j)\mathbf{f}(\mathbf{x}^j) \right)_i = (\mathbf{b}(\mathbf{x}^i))_i , \quad i = 1, \ldots, m, \tag{1.1}$$

with a known dense nonsymmetric matrix $\mathbf{k}$, an unknown source density $\mathbf{f}$, and the right-hand side $\mathbf{b}$. To find the source density, the linear system (1.1) has to be solved. For the solution with any iterative method, the matrix-vector multiplication have to be calculated. This calculation requires $\mathscr{O}(m^2)$ multiplications, but it can be accelerated if the matrix components $\mathbf{k}(\mathbf{x}^i, \mathbf{x}^j)$ can be represented approximately by

$$\mathbf{k}(\mathbf{x}^i \mathbf{x}^j) \approx \sum_{l=1}^{M} \mathbf{u}_l(\mathbf{x}^i)\mathbf{v}_l(\mathbf{x}^j). \tag{1.2}$$

In this case, the left-hand side of (1.1) has the form

$$\left(\sum_{j=1}^{m} \mathbf{k}(\mathbf{x}^i, \mathbf{x}^j)\mathbf{f}(\mathbf{x}^j) \right)_i \approx \left(\sum_{l=1}^{M} \mathbf{u}_l(\mathbf{x}^i) \sum_{j=1}^{m} \mathbf{v}_l(\mathbf{x}^j)\mathbf{f}(\mathbf{x}^j) \right)_i . \tag{1.3}$$

The calculation of (1.3) requires $\mathscr{O}(Mm)$ multiplications, which is much faster than $\mathscr{O}(m^2)$ for increasing m.

This idea is used by the adaptive fast multipole method, introduced by Greengard and Rokhlin [3, 25, 26]. To apply this method in our case, we develop a new representation for the discrete hydrodynamical double layer potential kernel in form of complex series. Then we extend the results of Greengard and Rokhlin for the Laplace equation from [25] to a system of equations: We derive for the new representation of the hydrodynamical double layer potential its multipole and Taylor expansions, the appropriate translation, rotation, and conversion operators, as well as the corresponding error estimates. Numerical tests illustrate the efficiency of the method due to the significant reduction of the computational complexity.

The outline of the paper is as follows. In Section 2, we introduce the Dirichlet problem for the homogeneous Stokes equations in the two-dimensional case, a hydrodynamical double layer potential and the corresponding system of boundary integral equations. Section 3 contains the derivation of the new complex representation of the hydrodynamical potential and provides the analytical preliminaries, which are necessary for the justification of the fast multipole method. A short description of the AFMM-algorithm and the results of the numerical tests are presented in Section 4. Finally, conclusions are given in Section 5.

2 Problem Statement

We consider the interior Dirichlet problem for the Stokes system:

$$\begin{aligned} &-\Delta \mathbf{u} + \nabla p = \mathbf{0} \quad \text{in } \ G, \\ &\nabla \cdot \mathbf{u} = 0 \quad \text{in } \ G, \\ &\mathbf{u} = \mathbf{b} \quad \text{on } \ \partial G. \end{aligned} \tag{2.1}$$

These equations are well known from hydrodynamics: the vector-valued function $\mathbf{u}$ represents the viscosity field and the scalar function p denotes the pressure of a viscous incompressible fluid with conservative external forces. The first equation in (2.1) means the conservation of momentum, the second one represents the conservation of mass. In the third equation, $\mathbf{b}$ is a prescribed value on the boundary ∂G satisfying the compatibility condition

$$\int\limits_{\partial G} \mathbf{b} \cdot \mathbf{n} \mathrm{do} = 0$$

with the outward unit normal vector $\mathbf{n}$ on ∂G.

Throughout the paper, we consider the two-dimensional case. We assume that the domain $G \subset \mathbb{R}^2$ is bounded and simply connected with the compact boundary $\partial G \in C^2$. The functions $\mathbf{u}$ and $\mathbf{b}$ are two-dimensional vector-valued functions in G and ∂G respectively, and the boundary value $\mathbf{b}$ is given continuously on ∂G, Δ denotes the Laplacian, ∇ the gradient, and $\nabla\cdot$ the divergence in $\mathbb{R}^2$.

From hydrodynamical potential theory, it is well known (cf. [18, 19, 24]) that the interior Dirichlet problem has a solution $\mathbf{u}$, p, where $\mathbf{u}$ is unique and p is unique up to an additive constant. This solution can be represented by a hydrodynamical double layer potential. In the case of the velocity field $\mathbf{u}$, this potential has the form

$$\mathbf{u}(\mathbf{x}) := D\mathbf{q}(\mathbf{x}) := \int\limits_{\partial G} D(\mathbf{x}, \mathbf{y})\mathbf{q}(\mathbf{y}) \mathrm{do}_{\mathbf{y}}, \quad \mathbf{x} \in G, \tag{2.2}$$

where the unknown source density $\mathbf{q}$ can be determined as a solution of a uniquely solvable the system of Fredholm boundary integral equations of the form

$$\frac{1}{2}\mathbf{q}(\mathbf{x}) + D\mathbf{q}(\mathbf{x}) - \mathbf{n}(\mathbf{x}) \int\limits_{\partial G} \mathbf{q}(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) \mathrm{do}_{\mathbf{y}} = \mathbf{b}(\mathbf{x}), \quad \mathbf{x} \in \partial G, \tag{2.3}$$

and $D(\mathbf{x}, \mathbf{y})$ is defined by

$$D_{ki}(\mathbf{x}, \mathbf{y}) := -\frac{1}{\pi} \frac{(x_k - y_k)(x_i - y_i)(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^4}, \quad k, i = 1, 2. \tag{2.4}$$

To find the function $\mathbf{q}$ numerically, we parameterize the equations with $\gamma : [0, 1] \longrightarrow \partial G$ and discretize the integrals using the Nyström method with step size $h = 1/m$ and $j = 1, \ldots, m$. Then

$$\frac{1}{2}\widetilde{\mathbf{q}}(ih) + h \sum_{i \neq j = 1}^{m} \widetilde{D_\gamma}(ih, jh)\, \widetilde{\mathbf{q}}(jh) + h\mathscr{K}\widetilde{\mathbf{q}}(jh) - h\widetilde{\mathbf{n}}(ih) \sum_{j=1}^{m} \widetilde{\mathbf{q}}(jh) \cdot \widetilde{\mathbf{n}_\gamma}(jh) = \widetilde{\mathbf{b}}(ih) \tag{2.5}$$

with

$$\widetilde{\mathbf{q}}(ih) := \mathbf{q}(\gamma(ih)), \quad \widetilde{\mathbf{b}}(ih) := \mathbf{b}(\gamma(ih)), \quad \widetilde{\mathbf{n}}(ih) := \mathbf{n}(\gamma(ih)), \quad \widetilde{\mathbf{n}_\gamma}(jh) := \mathbf{n}(\gamma(jh))|\dot{\gamma}(jh)|,$$

and $\widetilde{D_\gamma}(ih, jh) := D(\gamma(ih), \gamma(jh))|\dot{\gamma}(jh)|$. In (2.5), $\mathscr{K}$ denotes the curvature matrix and can be obtained for the known boundary. For the solution of (2.5) with any iterative method, the matrix-vector multiplication have to be calculated. For this calculation we use the adaptive fast multipole method due to Greengard and Rokhlin [3, 26]. It allows the rapid evaluation of the logarithmic potential and force fields in systems involving large numbers of particles. Rokhlin defined the central strategy of this method like that of clustering particles at various spatial lengths and computing interactions with other clusters, which are sufficiently far away by means of multipole expansions. The interactions between particles near by are computed directly. This algorithm has an asymptotic CPU time estimate of $\mathscr{O}(m \log_2 m)$, where m is the number of discretization points in the simulation, and it does not depend on the distribution of these points (cf. [1]).

3 Realization of the Hydrodynamical Double Layer Potential by the Fast Multipole Method

To apply the adaptive fast multipole method to (2.5) in a similar way as it was done by Greengard and Rokhlin in [3], the complex representation of the double layer tensor (2.4) has to be found.

Let $z, z_0, z_1, \ldots, z_m$ be different points in the complex plane. For $\mathbf{x}, \mathbf{y}, \mathbf{n} \in \mathbb{R}^2$ we set

$$\begin{aligned} \mathbf{x} &= (x_1, x_2) \leftrightarrow z = x_1 + i x_2, \\ \mathbf{y} &= (y_1, y_2) \leftrightarrow z_0 = y_1 + i\, y_2, \\ \mathbf{n} &= (n_1, n_2) \leftrightarrow N = n_1 + i\, n_2, \end{aligned} \tag{3.1}$$

where $N := n_1 + in_2$ is the complex version of $\mathbf{n}(\mathbf{y})$. We denote by $\overline{N} := n_1 - in_2$ its conjugate, and set $M := n_2 + in_1$ and $\overline{M} := n_2 - in_1$. We denote by $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the real and imaginary part of $z \in \mathbb{C}$ respectively. With this notation, we get the following representations:

$$\begin{aligned} \operatorname{Re} \frac{N}{z - z_0} &= \frac{|z-z_0|^2}{|z-z_0|^2} \operatorname{Re} \frac{(n_1 + i\, n_2)((x_1 - y_1) - i\,(x_2 - y_2))}{|z-z_0|^2} \\ &= \frac{(x_1 - y_1)^2 + (x_2 - y_2)^2}{|z-z_0|^2} \frac{(n_1\,(x_1 - y_1) + n_2\,(x_2 - y_2))}{|z-z_0|^2} \\ &= \frac{n_1\,(x_1 - y_1)^3 + n_1\,(x_1 - y_1)(x_2 - y_2)^2}{|z-z_0|^4} + \frac{n_2\,(x_1 - y_1)^2 (x_2 - y_2) + n_2\,(x_2 - y_2)^3}{|z-z_0|^4}, \end{aligned}$$

$$\begin{aligned} \operatorname{Re} \frac{N}{(z - z_0)^2} &= \operatorname{Re} \frac{(n_1 + i\, n_2)((x_1 - y_1) - i\,(x_2 - y_2))^2}{|z-z_0|^4} \\ &= \frac{n_1\,(x_1 - y_1)^2 - n_1\,(x_2 - y_2)^2 + 2n_2 (x_1 - y_1)(x_2 - y_2)}{|z-z_0|^4}, \end{aligned}$$

$$\operatorname{Re} \frac{\overline{N}}{(z - z_0)^2} = \frac{n_1 (x_1 - y_1)^2 - n_1\,(x_2 - y_2)^2 - 2n_2 (x_1 - y_1)(x_2 - y_2)}{|z-z_0|^4},$$

as well as

$$\frac{1}{4} \operatorname{Re}\,(z - z_0) \left(\operatorname{Re} \frac{N}{(z - z_0)^2} - \operatorname{Re} \frac{\overline{N}}{(z - z_0)^2} \right) = \frac{n_2\,(x_1 - y_1)^2 (x_2 - y_2)}{|z-z_0|^4}$$

and

$$\frac{1}{4}\,\mathrm{Im}\,(z-z_0)\,\left(\mathrm{Re}\,\frac{M}{(z-z_0)^2}-\mathrm{Re}\,\frac{\overline{M}}{(z-z_0)^2}\right)=\frac{n_1\,(x_1-y_1)(x_2-y_2)^2}{|z-z_0|^4}.$$

By an elementary but lengthy calculations, we obtain the complex representation of the components of the tensor in (2.4) as follows:

$$D_{11}=-\frac{1}{\pi}\left(\mathrm{Re}\,\frac{N}{z-z_0}-\frac{1}{4}\,\mathrm{Re}\,(z-z_0)\left(\mathrm{Re}\,\frac{N}{(z-z_0)^2}\,\mathrm{Re}\,\frac{\overline{N}}{(z-z_0)^2}\right)\right.$$
$$\left.+\frac{1}{4}\,\mathrm{Im}\,(z-z_0)\left(3\,\mathrm{Re}\,\frac{\overline{M}}{(z-z_0)^2}+\mathrm{Re}\,\frac{M}{(z-z_0)^2}\right)\right),$$

$$D_{12}=D_{21}=-\frac{1}{\pi}\left(\frac{1}{4}\,\mathrm{Re}\,(z-z_0)\left(\mathrm{Re}\,\frac{M}{(z-z_0)^2}-\mathrm{Re}\,\frac{\overline{M}}{(z-z_0)^2}\right)\right.$$
$$\left.+\frac{1}{4}\,\mathrm{Im}\,(z-z_0)\left(\mathrm{Re}\,\frac{N}{(z-z_0)^2}-\mathrm{Re}\,\frac{\overline{N}}{(z-z_0)^2}\right)\right),$$

$$D_{22}=-\frac{1}{\pi}\left(\mathrm{Re}\,\frac{N}{z-z_0}-\frac{1}{4}\,\mathrm{Re}\,(z-z_0)\left(3\,\mathrm{Re}\,\frac{N}{(z-z_0)^2}+\mathrm{Re}\,\frac{\overline{N}}{(z-z_0)^2}\right)\right.$$
$$\left.\frac{1}{4}\,\mathrm{Im}\,(z-z_0)\left(\mathrm{Re}\,\frac{M}{(z-z_0)^2}-\mathrm{Re}\,\frac{\overline{M}}{(z-z_0)^2}\right)\right).$$

The most costly part by the numerical solution of the linear system (2.5) is the evaluation of the sum

$$\sum_{i\neq j=1}^{m}\widetilde{D_\gamma}(ih,jh)\,\widetilde{\mathbf{q}}(jh),\tag{3.2}$$

i.e., the calculation of the finite sums of the type

$$\sum_{j=1}^{m}\mathrm{Re}\,(z-z_j)\;\;\mathrm{Re}\,\frac{q_jN_j}{(z-z_j)^2}.\tag{3.3}$$

Here, the fast multipole method can be used.

We define the functions $\Phi_{z_0,N}$ and Φ by

$$\Phi_{z_0,N}\,(z):=\mathrm{Re}\,(z-z_0)\,\mathrm{Re}\,\frac{q\,N}{(z-z_0)^2},\tag{3.4}$$

$$\Phi(z):=\sum_{j=1}^{m}\Phi_{z_j,N_j}(z):=\sum_{j=1}^{m}\mathrm{Re}(z-z_j)\,\mathrm{Re}\,\frac{q_jN_j}{(z-z_j)^2}\tag{3.5}$$

and obtain the following assertion for the multipole expansion for the function $\Phi_{z_0,N}$.

Lemma 3.1. *Let $q\in\mathbb{R}$ and $N,z_0\in\mathbb{C}$ be given. Then from* (3.4) *we obtain the following representation for $\Phi_{z_0,N}$:*

$$\Phi_{z_0,N}\,(z)=qN\,\mathrm{Re}\;z\,\mathrm{Re}\sum_{k=0}^{\infty}(k+1)\frac{z_0^k}{z^{k+2}}-qN\,\mathrm{Re}\sum_{k=0}^{\infty}(k+1)\frac{z_0^k\,\mathrm{Re}\;z_0}{z^{k+2}}\tag{3.6}$$

for any $z \in \mathbb{C}$ *with* $|z| > |z_0|$.

Proof. Note that $|z_0/z| < 1$. The assertion follows from the expansion

$$\frac{1}{(z-z_0)^2} = \frac{1}{z^2}\left(1+\frac{z_0}{z}+\ldots+\left(\frac{z_0}{z}\right)^n+\ldots\right)^2 = \frac{1}{z^2}\left(\sum_{k=0}^{\infty}\frac{z_0^k}{z^k}\right)^2 = \sum_{k=0}^{\infty}(k+1)\,\frac{z_0^k}{z^{k+2}}. \qquad \square$$

The following theorem presents the multipole expansion of $\Phi(z)$ from (3.5) and the corresponding error estimate.

Theorem 3.2. *Let* $m \in \mathbb{N}$, $q_j, r \in \mathbb{R}$ *and* $N_j, z_j \in \mathbb{C}$, $j = 1, \ldots, m$, *be given. Then for any* $z \in \mathbb{C}$ *with* $|z - z_j| > r$ *(Fig. 1) the function* $\Phi(z)$ *from* (3.5) *is given by the sum of the multipole expansions*

$$\Phi(z) = \mathrm{Re}\ z\ \mathrm{Re}\sum_{k=1}^{\infty}\frac{a_k}{z^{k+1}} - \mathrm{Re}\sum_{k=1}^{\infty}\frac{a'_k}{z^{k+1}} \tag{3.7}$$

with

$$\begin{aligned} a_k &= k\sum_{j=1}^{m} q_j N_j z_j^{k-1}, \\ a'_k &= k\sum_{j=1}^{m} q_j N_j z_j^{k-1}\,\mathrm{Re}\ z_j. \end{aligned} \tag{3.8}$$

Furthermore, for any $p \in \mathbb{N}$

$$\begin{aligned} \left|\sum_{k=p+1}^{\infty}\frac{a_k}{z^{k+1}}\right| &\leqslant \alpha_1\left|\frac{r}{z}\right|^p, \\ \left|\sum_{k=p+1}^{\infty}\frac{a'_k}{z^{k+1}}\right| &\leqslant \alpha_2\left|\frac{r}{z}\right|^p \end{aligned} \tag{3.9}$$

with

$$\begin{aligned} \alpha_i &= \frac{A_i\big(r+(p+1)(|z|-r)\big)}{|z|\,(|z|-r)^2}, \quad i = 1, 2, \\ A_1 &= \sum_{j=1}^{m}|q_j N_j|, \quad A_2 = \sum_{j=1}^{m}|q_j N_j\,\mathrm{Re}\ z_j|. \end{aligned} \tag{3.10}$$

Proof. With an easy calculation, we find

$$\Phi(z) = \sum_{j=1}^{m}\mathrm{Re}\ z\,\mathrm{Re}\,\frac{q_j N_j}{(z-z_j)^2} - \sum_{j=1}^{m}\mathrm{Re}\ z_j\,\mathrm{Re}\,\frac{q_j N_j}{(z-z_j)^2} =: T_1(z) - T_2(z),$$

where

$$T_1(z) = \sum_{j=1}^{m}\mathrm{Re}\ z\,\mathrm{Re}\,\frac{q_j N_j}{(z-z_j)^2} \overset{(3.6)}{=} \mathrm{Re}\ z\,\mathrm{Re}\left(\sum_{j=1}^{m}\frac{q_j N_j}{z^2}\sum_{k=1}^{\infty}k\,\frac{z_j^{k-1}}{z^{k-1}}\right) = \mathrm{Re}\ z\,\mathrm{Re}\left(\sum_{k=1}^{\infty}\frac{a_k}{z^{k+1}}\right),$$

$$a_k := k\sum_{j=1}^{m} q_j N_j z_j^{k-1}$$

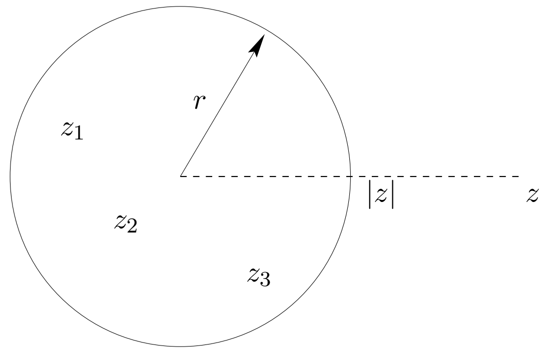

Figure 1: Multipole expansion.

and

$$T_2(z) = \sum_{j=1}^{m} \operatorname{Re}\, z_j \operatorname{Re}\, \frac{q_j N_j}{(z - z_j)^2} = \sum_{j=1}^{m} \operatorname{Re}\, \frac{q_j N_j \operatorname{Re}\, z_j}{(z - z_j)^2} = \operatorname{Re}\left(\sum_{k=1}^{\infty} \frac{a'_k}{z^{k+1}} \right),$$

$$a'_k := k \sum_{j=1}^{m} q_j N_j z_j^{k-1} \operatorname{Re}\, z_j.$$

This leads to

$$\left| \Phi(z) - \left(\operatorname{Re}\, z \operatorname{Re} \sum_{k=1}^{p} \frac{a_k}{z^{k+1}} - \operatorname{Re} \sum_{k=1}^{p} \frac{a'_k}{z^{k+1}} \right) \right|$$

$$= \left| \operatorname{Re}\, z \operatorname{Re} \sum_{k=p+1}^{\infty} \frac{a_k}{z^{k+1}} - \operatorname{Re} \sum_{k=p+1}^{\infty} \frac{a'_k}{z^{k+1}} \right| \leqslant |\operatorname{Re}\, z| \left| \operatorname{Re} \sum_{k=p+1}^{\infty} \frac{a_k}{z^{k+1}} \right| + \left| \operatorname{Re} \sum_{k=p+1}^{\infty} \frac{a'_k}{z^{k+1}} \right|. \quad (3.11)$$

With the notation from (3.10), the series in the first term of (3.11) can be estimated by

$$\left| \sum_{k=p+1}^{\infty} \frac{a_k}{z^{k+1}} \right| \leqslant \left| \frac{A_1}{z^2} \sum_{k=p+1}^{\infty} \frac{k\, r^{k-1}}{z^{k-1}} \right| = \frac{A_1}{|z|^2} \left| \frac{r}{z} \right|^p \sum_{k=0}^{\infty} (k + p + 1) \left(\frac{r}{|z|} \right)^k = \alpha_1 \left| \frac{r}{z} \right|^p$$

with

$$\alpha_1 = \frac{A_1 \big(r + (p+1)(|z| - r) \big)}{|z|\, (|z| - r)^2}.$$

Analogously, for the series in the second term of (3.11) we obtain the estimate

$$\left| \sum_{k=p+1}^{\infty} \frac{a'_k}{z^{k+1}} \right| \leqslant \alpha_2 \left| \frac{r}{z} \right|^p$$

with

$$\alpha_2 = \frac{A_2 \big(r + (p+1)(|z| - r) \big)}{|z|\, (|z| - r)^2}. \qquad \square$$

The following theorems describe translation operators for the multipole expansion allowing us to manipulate this expansion in a manner required by the adaptive fast multipole algorithm. The next theorem provides an operator for shifting the center of a multipole expansion.

Theorem 3.3. *Let $m \in \mathbb{N}$, $R, q_j \in \mathbb{R}$ and $N_j, z_0 \in \mathbb{C}$, $j = 0, \ldots, m$, be given. Furthermore, for any $z \in \mathbb{C}$ with $|z - z_0| < R$ (Fig. 2)*

$$\widetilde{\Phi}(z) := \sum_{k=1}^{\infty} \frac{a_k}{(z - z_0)^{k+1}}, \tag{3.12}$$

where $a_k \in \mathbb{C}$ are given in (3.8). Then for any $z \in \mathbb{C}$ with $|z| > R' := R + |z_0|$

$$\widetilde{\Phi}(z) = \sum_{l=1}^{\infty} \frac{b_l}{z^{l+1}}, \qquad b_l = \sum_{k=1}^{l} \binom{l}{k} a_k\, z_0^{l-k} \tag{3.13}$$

and for all $p \in \mathbb{N}$

$$\left| \widetilde{\Phi}(z) - \sum_{l=1}^{p} \frac{b_l}{z^{l+1}} \right| \leqslant \alpha \left| \frac{R'}{z} \right|^{p+1} \tag{3.14}$$

with

$$\alpha = \frac{A\big(R' + (p+1)(|z| - R')\big)}{|z|(|z| - R')^2}, \quad A = \sum_{j=1}^{m} |q_j N_j|.$$

Proof. For $|z| > |z_0|$ we obtain

$$\frac{a_k}{(z - z_0)^{k+1}} = \frac{a_k}{z^{k+1}} + \frac{(k+1)\, a_k}{z^{k+2}}\, z_0 + \ldots + \binom{l+k}{l} \frac{a_k}{z^{l+k+1}}\, z_0^l + \ldots. \tag{3.15}$$

Hence

$$\begin{aligned}
\widetilde{\Phi}(z) &= \frac{a_1}{z^2} + \ldots + \frac{\binom{l}{l-1} a_1 z_0^{l-1} + \binom{l}{l-2} a_2 z_0^{l-2} + \ldots + \binom{l}{l-k} a_k z_0^{l-k} + \ldots + a_l}{z^{l+1}} + \ldots \\
&= \sum_{l=1}^{\infty} \frac{1}{z^{l+1}} \sum_{k=1}^{l} \binom{l}{k} a_k\, z_0^{l-k} = \sum_{l=1}^{\infty} \frac{b_l}{z^{l+1}}
\end{aligned}$$

with

$$b_l = \sum_{k=1}^{l} \binom{l}{k} a_k\, z_0^{l-k}.$$

Moreover, for $a_k = k \sum_{j=1}^{m} q_j N_j (z_j - z_0)^{k-1}$ we find

$$\begin{aligned}
\left| \widetilde{\Phi}(z) - \sum_{l=1}^{p} \frac{b_l}{z^{l+1}} \right| &= \left| \sum_{l=p+1}^{\infty} \frac{\sum\limits_{k=1}^{l} \big(k \sum\limits_{i=1}^{m} q_i\, N_i\, (z_i - z_0)^{k-1}\big) z_0^{l-k} \binom{l}{k}}{z^{l+1}} \right| \\
&\leqslant \left| \sum_{l=p+1}^{\infty} \frac{\sum\limits_{k=1}^{l} k\, R^{k-1} \big(\sum\limits_{i=1}^{m} q_i N_i \big) z_0^{l-k} \binom{l}{k}}{z^{l+1}} \right| \\
&= \left| \sum_{l=p+1}^{\infty} \frac{A\,(R + z_0)^{l-1}\, l}{z^{l+1}} \right| \leqslant \frac{A}{|z|^2} \left| \frac{R'}{z} \right|^p \sum_{l=0}^{\infty} (l + p + 1) \left(\frac{R'}{|z|} \right)^l = \alpha \left| \frac{R'}{z} \right|^p,
\end{aligned}$$

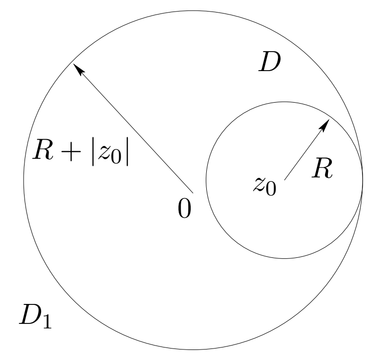

Figure 2: Shifting the center of the multipole expansion.

with
$$\alpha=\frac{A\big(R'+(p+1)(|z|-R')\big)}{|z|\,(|z|-R')^2},\qquad A=\sum_{i=1}^{m}|q_jN_j|.\qquad\square$$

The following theorem describes, how to convert the shifted expansion into a local (Taylor) expansion in a circular region of analyticity.

Theorem 3.4. *Let $m\in\mathbb{N}$, $c,\mathbb{R}q_j\in\mathbb{R}$ and $N_j,z_0\in\mathbb{C}$, $j=1,\dots,m$, be given with $|z_0|>(c+1)R$ and $|z-z_0|<R$ for all $z\in\mathbb{C}$. Then the multipole expansion $\widetilde{\Phi}\,(z)$ from* (3.12) *converges in the interior of the circle of radius R with center at the origin* (*circle D_2 in Fig.* 3) *and can be represented by*

$$\widetilde{\Phi}\,(z)=\sum_{l=0}^{\infty}b_lz^l,\qquad b_l=\frac{1}{z_0^{l+1}}\sum_{k=1}^{\infty}(-1)\binom{l+k}{k}\frac{a_k}{(-z_0)^k}.\tag{3.16}$$

Furthermore, for any $p\in\mathbb{N}$, $p\geqslant\max\big(2,2c/(c-1)\big)$, an error bound for the truncated series is given by

$$\left|\widetilde{\Phi}\,(z)-\sum_{l=0}^{p}b_l\,z^l\right|\leqslant\frac{4\,A\,e\,p^2\,(c+1)}{R^2\,(p+c-1)^2\,(c-1)}\left(\frac{1}{c}\right)^{p+1},\tag{3.17}$$

where e is the Euler constant and A is defined by (3.14).

Proof. For the multipole expansion of

$$\widetilde{\Phi}\,(z)=\sum_{k=1}^{\infty}\frac{a_k}{(z-z_0)^{k+1}}$$

we obtain

$$\begin{aligned}\widetilde{\Phi}(z)&=+\frac{a_1}{z_0^2}+\frac{2\,a_1}{z_0^3}z+\ldots+\binom{l+1}{l}\frac{a_1}{z_0^{l+2}}\,z^l+\ldots-\frac{a_2}{z_0^3}-\frac{3a_2}{z_0^4}\,z-\ldots-\binom{l+2}{l}\frac{a_2}{z_0^{l+3}}\,z^l-\ldots\\&=\sum_{l=0}^{\infty}z^l\sum_{k=1}^{\infty}\binom{l+k}{k}\frac{a_k}{z_0^{l+k+1}}(-1)^{k+1}=\sum_{l=0}^{\infty}b_lz^l\end{aligned}$$

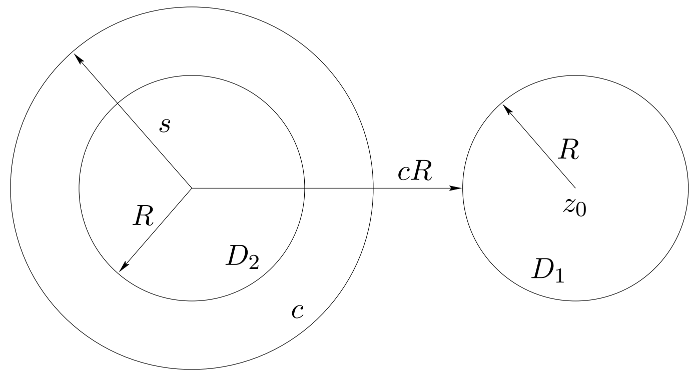

Figure 3: Conversion of the multipole expansion into a Taylor expansion.

with

$$b_l = \frac{1}{z_0^{l+1}} \sum_{k=1}^{\infty} (-1) \binom{l+k}{k} \frac{a_k}{(-z_0)^k}.$$

To prove the approximation (3.17), assume that C is a circle of radius s with $s = cR(p-1)/p$. Then for any $p \geqslant 2c/(c-1)$

$$R < \frac{cR+R}{2} < s < cR, \tag{3.18}$$

and we get

$$\left| \widetilde{\Phi}\,(z) - \sum_{l=0}^{p} b_l\, z^l \right| = \left| \sum_{l=p+1}^{\infty} b_l\, z^l \right| \leqslant \left(\frac{|z|}{s}\right)^{p+1} M \Bigg/ \left(1 - \frac{|z|}{s}\right) =: F \tag{3.19}$$

with

$$M = \max_{t \in C} |\widetilde{\Phi}(t)|.$$

For $t \in C$ we have $|t - z_0| \geqslant cR + R - s$. By (3.8) and (3.10), we get $|a_k| \leqslant k\, A\, R^{k-1}$, and it follows with $s = cR(p-1)/p$:

$$\begin{aligned} |\widetilde{\Phi}(t)| &\leqslant \sum_{k=1}^{\infty} \frac{|a_k|}{|t - z_0|^{k+1}} \leqslant A \sum_{k=1}^{\infty} \frac{k\, R^{k-1}}{(cR + R - s)^{k+1}} \\ &= \frac{A}{(cR+R-s)^2} \sum_{k=0}^{\infty} (k+1) \left(\frac{R}{cR+R-s}\right)^k = \frac{A\, p^2}{R^2\,(p+c-1)^2}. \end{aligned}$$

Then we can estimate (3.19) by

$$\begin{aligned} F &\leqslant \left(\frac{|z|}{s}\right)^{p+1} \frac{A\, p^2}{R^2\,(p+c-1)^2} \Bigg/ \left(1 - \frac{|z|}{s}\right) \\ &\leqslant \left(\frac{R}{cR\frac{p-1}{p}}\right)^{p+1} \frac{Ap^2}{R^2(p+c-1)^2} \frac{cR+R}{cR-R} \leqslant \left(\frac{1}{c}\right)^{p+1} \frac{4Aep^2(c+1)}{R^2(p+c-1)^2(c-1)}, \qquad p \geqslant 2. \quad \square \end{aligned}$$

The next lemma provides a formula for shifting the center of a local (Taylor) expansion.

Lemma 3.5. *For any* $z, z_0, a_k \in \mathbb{C}$ $(k = 0, 1, \ldots, n)$,

$$\sum_{k=0}^{n} a_k (z - z_0)^k = \sum_{l=0}^{n} \left(\sum_{k=l}^{n} a_k \binom{k}{l} (-z_0)^{k-l} \right) z^l. \tag{3.20}$$

Proof. The assertion immediately follows from

$$\sum_{k=0}^{n} a_k (z - z_0)^k = \sum_{k=0}^{n} a_k \sum_{m=0}^{k} \binom{k}{m} (-z_0)^m z^{k-m} = \sum_{l=0}^{n} \left(\sum_{k=l}^{n} a_k \binom{k}{l} (-z_0)^{k-l} \right) z^l. \qquad \square$$

4 The Adaptive Fast Multipole Algorithm

4.1 General idea

The analytical tools presented in lemmas and theorems of Section 3 enable us now to use the adaptive fast multipole method for calculating the matrix-vector products by the numerical solution of (2.3):

Let m discretization points in some square (computational box) in $\mathbb{R}^2$ and some integer $s \ll m$ be given. The AFMM-algorithm starts with the refinement of the computational box into smaller and smaller squares, obtaining the mesh level $l + 1$ from the level l by subdividing those squares of level l that contain more than s discretization points (fathers) into four squares of the same size (sons). We distinguish the near field squares from the far field squares and compute the sum $\sum\limits_{i \neq j = 1}^{m} \widetilde{D_\gamma}(ih, jh)\, \widetilde{\mathbf{q}}(jh)$ from (3.2) for the discretization points in the near field directly. For the far field we use the analytical tools from Section 3 and proceed as following:

- *Step* 1. For each childless box b we combine its discretization points, and calculate the coefficients of the multipole expansions using Theorem 3.2.
- *Step* 2. All those contributions will be added by means of Theorem 3.3 to the father (one level up in the hierarchy).
- *Step* 3. The "transfers" of the contributions of the fathers' neighbors will be performed with Theorem 3.4.
- *Step* 4. Using Lemma 3.5, all those contributions will be added to the sons.
- *Step* 5. We sum all calculated contributions and add the near field part to obtain the final value of the matrix-vector product in some point.

4.2 Numerical tests

We assume a constant density $\mathbf{q} = 1$ in the hydrodynamical double layer potential $D\mathbf{q}(\mathbf{x})$ and compute the hydrodynamical double layer potential in all discretization points on the boundary using two methods: The direct matrix-vector multiplication and adaptive fast multipole method briefly presented in Subsection 4.1. We consider a bounded ball like domain G with the boundary defined by

$$x(t) := \cos(2\pi\, t),$$
$$y(t) := \begin{cases} \sin(2\pi\, t), & t \in [0, 1/8] \cup [3/8, 1], \\ -\dfrac{\sqrt{2}}{4}\left(\cos^4(2\pi t) + cos^2(2\pi t) - \dfrac{11}{4}\right), & t \in]1/8, 3/8[, \end{cases} \tag{4.1}$$

where $t \in [0, 1]$. We discretize the boundary ∂G by m discretization points $(x(t_i), y(t_i))$ with $t_i := i/m$, $i = 1, \ldots, m$. In the following test, we use the value $s = 50$ for the maximum number of discretization points in a childless box, and set $p = 25$ for the number of terms in the expansions. Table 4.1 provide the information about the CPU time (in seconds) and the storage (in MB) costs. Here, l denotes the number of refinements, t_{dir} and t_{AFMM} is the CPU time in seconds for the direct matrix-vector multiplication and for the adaptive fast multipole method respectively. In the last column, we present the information about the additional storage that we need for the adaptive fast multipole method in compare to the direct method.

Table 4.1. Performance of the direct matrix-vector multiplication vs. AFMM by the calculation of the hydrodynamical double layer potential

m	l	t_{dir}, sec	t_{AFMM}, sec	additional storage, MB
4000	5	4.42	3.99	0.31
8000	6	17.65	8.80	0.15
16000	7	82.75	17.71	-1.41
32000	8	391.20	38.20	-10.00
64000	9	1718.30	84.00	-49.33

The results show that by the adaptive fast multipole method, in contrast to the direct method, the CPU time grows almost linearly with the number of discretization points. Hence the adaptive fast multipole method essentially reduces the costs with respect to time and, in addition, with respect to storage for large m.

5 Conclusions

In this paper, we have presented the theoretical results, which establish and justify the possibility of usage of the adaptive fast multipole method for the rapid solution of the interior Dirichlet problem of the Stokes equations in the two dimensional case. Therefore, we found a new complex representation of the hydrodynamical double layer potential and provided for this representation statements about its multipole and Taylor expansions, as well as the appropriate error estimates, which are necessary for the implementation of the adaptive fast multipole method. The numerical tests demonstrated efficiency of the calculations performed on the basis of the presented theory. Especially, the almost linear complexity by the calculation of matrix-vector products has been achieved. The corresponding investigations for the exterior boundary value problems of the two-dimensional Stokes equations will be reported at a later date.

The generalization of the presented results to the practical important three-dimensional case of the Stokes equations (in this case, the multipole expansion will be obtained in the terms of spherical harmonics) is the object of the forthcoming work.

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